

# Generalising the Willmore equation: submanifold conformal invariants from a boundary Yamabe problem

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## Abstract

The Willmore energy, alias bending energy or rigid string action, and its variation—the *Willmore invariant*—are important surface conformal invariants with applications ranging from cell membranes to the entanglement entropy in quantum gravity. In work of Andersson, Chruściel, and Friedrich, the same invariant arises as the obstruction to smooth boundary asymptotics to the Yamabe problem of finding a metric in a conformal class with constant scalar curvature. We use conformal geometry tools to describe and compute the asymptotics of the Yamabe problem on a conformally compact manifold and thus produce higher order hypersurface conformal invariants that generalise the Willmore invariant. We give a holographic formula for these as well as variational principles for the lowest lying examples.

## 1 Introduction

While much is known about the invariants of conformal manifolds, the same cannot be said for the invariants of submanifolds in conformal geometries. Codimension-1 embedded submanifolds (or *hypersurfaces*) are important for applications in geometric analysis and physics. An extremely interesting example is the Willmore equation

$$(1) \quad \bar{\Delta}H + 2H(H^2 - K) = 0,$$

for an embedded surface  $\Sigma$  in Euclidean 3-space  $\mathbb{E}^3$ . Here  $H$  and  $K$  are, respectively, the mean and Gauß curvatures, while  $\bar{\Delta}$  is the Laplacian induced on  $\Sigma$ . We shall term the left hand side of this equation the *Willmore invariant*; as given, this quantity is invariant under Möbius transformations of the ambient  $\mathbb{E}^3$ . A key feature is the linearity of its highest order term,  $\bar{\Delta}H$ . This linearity is important for PDE problems, but also means that the Willmore invariant should be viewed as a fundamental curvature quantity.

In the 1992 article [1], Andersson, Chruściel and Friedrich (ACF) (building on the works [14, 3, 4]) identified a conformal surface invariant that obstructs smooth boundary asymptotics for a Yamabe solution on a conformally compact 3-manifold (and gave some information on the obstructions in dimension  $n + 1 = d > 3$ ). It is straightforward

to show that this invariant is the same as that arising from the variation of the Willmore energy; in particular its specialisation to surfaces in  $\mathbb{E}^3$  agrees with (1). We show how tools from conformal geometry can be used to describe and compute the asymptotics of the Yamabe problem on a conformally compact manifold. This reveals higher order hypersurface conformal invariants that generalise the curvature obstruction found by ACF. In particular, for hypersurfaces of arbitrary even dimension this yields higher order conformally invariant analogues of the usual Willmore equation on surfaces in 3-space. The construction also leads to a general theory for constructing and treating conformal hypersurface invariants along the lines of holography and the Fefferman-Graham programme for constructing invariants of a conformal structure via their Poincaré-Einstein and “ambient” metrics [6]. In this announcement, we focus only on main results, a detailed account of this general theory will be presented elsewhere [10].

## 2 The problem

Given a Riemannian  $d$ -manifold  $(M, g)$  with boundary  $\Sigma := \partial M$ , one may ask whether there is a smooth real-valued function  $u$  on  $M$  satisfying the following two conditions:

- (1)  $u$  is a defining function for  $\Sigma$  (i.e.,  $\Sigma$  is the zero set of  $u$ , and  $du_x \neq 0 \ \forall x \in \Sigma$ );
- (2)  $\bar{g} := u^{-2}g$  has scalar curvature  $\text{Sc}^{\bar{g}} = -d(d-1)$ .

Here  $d$  is the exterior derivative. We assume  $d \geq 3$  and all structures are  $C^\infty$ .

Assuming  $u > 0$  and setting  $u = \rho^{-2/(d-2)}$ , part (2) of this problem gives the Yamabe equation. The problem fits nicely into the framework of conformal geometry: Recall that a conformal structure  $\mathbf{c}$  on a manifold is an equivalence class of metrics where the equivalence relation  $\hat{g} \sim g$  means that  $\hat{g} = \Omega^2 g$  for some positive function  $\Omega$ . The line bundle  $(\Lambda^d T^*M)^2$  is oriented and for  $w \in \mathbb{R}$  the bundle of *conformal densities* of weight  $w$ , denoted  $\mathcal{E}[w]$ , is defined to be the oriented  $\frac{w}{2d}$ -root of this (we use the same notation for bundles as for their smooth section spaces). Locally each  $g \in \mathbf{c}$  determines a volume form and, squaring this, globally a section of  $(\Lambda^d T^*M)^2$ . So, on a conformal manifold  $(M, \mathbf{c})$  there is a canonical section  $\mathbf{g}$  of  $S^2 T^*M \otimes \mathcal{E}[2]$  called the conformal metric. Thus each metric  $g \in \mathbf{c}$  is naturally in 1 : 1 correspondence with a (strictly) positive section  $\tau$  of  $\mathcal{E}[1]$  via  $g = \tau^{-2}\mathbf{g}$ . Also, the Levi-Civita connection  $\nabla$  of  $g$  preserves  $\tau$ , and hence  $\mathbf{g}$ . Thus we are led to the conformally invariant equation on a weight 1 density  $\sigma \in \mathcal{E}[1]$

$$(2) \quad S(\sigma) := (\nabla \sigma)^2 - \frac{2}{d} \sigma \left( \Delta + \frac{\text{Sc}}{2(d-1)} \right) \sigma = 1,$$

where  $\mathbf{g}$  and its inverse are used to raise and lower indices,  $\Delta = \mathbf{g}^{ab} \nabla_a \nabla_b$  and  $\text{Sc}$  means  $\mathbf{g}^{bd} R_{ab}{}^a{}_d$ , with  $R$  the Riemann tensor. Choosing  $\mathbf{c} \ni g = \tau^{-2}\mathbf{g}$ , equation (2) becomes exactly the PDE obeyed by the smooth function  $u = \sigma/\tau$  solving part (2) of the problem above. Since  $u$  is a defining function this means  $\sigma$  is a *defining density* for  $\Sigma$ , meaning that it is a section of  $\mathcal{E}[1]$ , its zero locus  $\mathcal{Z}(\sigma) = \Sigma$ , and  $\nabla \sigma_x \neq 0 \ \forall x \in \Sigma$ . For our purpose we only need to treat the problem formally (so it applies to any hypersurface):

**Problem 1.** Let  $\Sigma$  be an embedded hypersurface in a conformal manifold  $(M, \mathbf{c})$  with  $d \geq 3$ . Given a defining density  $\sigma$  for  $\Sigma$ , find a new, smooth, defining density  $\bar{\sigma}$  such that

$$(3) \quad S(\bar{\sigma}) = 1 + \bar{\sigma}^\ell A_\ell,$$

for some  $A_\ell \in \mathcal{E}[-\ell]$ , where  $\ell \in \mathbb{N} \cup \infty$  is as high as possible.

### 3 The main results

Here we use the notation  $\mathcal{O}(\sigma^\ell)$  to mean plus  $\sigma^\ell A$  for some smooth  $A \in \mathcal{E}[-\ell]$ .

**Theorem 2.** *Let  $\Sigma$  be an oriented embedded hypersurface in  $(M, \mathbf{c})$ , where  $d \geq 3$ , then:*

- *There is a distinguished defining density  $\bar{\sigma} \in \mathcal{E}[1]$  for  $\Sigma$ , unique to  $\mathcal{O}(\bar{\sigma}^{d+1})$ , such that*

$$(4) \quad S(\bar{\sigma}) = 1 + \bar{\sigma}^d B_{\bar{\sigma}},$$

where  $B_{\bar{\sigma}} \in \mathcal{E}[-d]$  is smooth on  $M$ . Given any defining density  $\sigma$ , then  $\bar{\sigma}$  depends smoothly on  $(M, \mathbf{c}, \sigma)$  via a canonical formula  $\bar{\sigma}(\sigma)$ .

- $\mathcal{B} := B_{\bar{\sigma}(\sigma)}|_{\Sigma}$  is independent of  $\sigma$  and is a natural invariant determined by  $(M, \mathbf{c}, \Sigma)$ .

For any unit conformal defining density  $\bar{\sigma}$  satisfying Eq. (4) of the Theorem, it is straightforward, although tedious, to calculate  $\mathcal{B}$ . For  $d = 3$  we obtain

$$(5) \quad \mathcal{B} = 2(\bar{\nabla}_{(i} \bar{\nabla}_{j)\circ} + H \mathring{\Pi}_{ij} + R_{(ij)\circ}^\top) \mathring{\Pi}^{ij},$$

where  $\mathring{\Pi}_{ij}$  is the trace-free part of the second fundamental form  $\Pi_{ij}$ ,  $R_{(ij)\circ}^\top$  is the trace-free part of the projection of the ambient Ricci tensor along  $\Sigma$ , and  $\bar{\nabla}$  is the Levi-Civita for the metric on  $\Sigma$  induced by  $g$ . Equation (5) agrees with [1, Theorem 1.3] and [7] and, by using the Gauß–Codazzi equations, agrees with (1) for  $\Sigma$  in  $\mathbb{E}^3$ . (We note that Eq. (4) is consistent with [1, Lemma 2.1].)

For  $d = 4$  and (specializing to) conformally flat structures  $\mathbf{c}$ , evaluated on  $g \in \mathbf{c}$  with  $g$  flat, our result for the obstruction density  $\mathcal{B}$  of Theorem 2 is

$$(6) \quad \mathcal{B} = \frac{1}{6} \left( (\bar{\nabla}_k \mathring{\Pi}_{ij}) (\bar{\nabla}^k \mathring{\Pi}^{ij}) + 2 \mathring{\Pi}^{ij} \bar{\Delta} \mathring{\Pi}_{ij} + \frac{3}{2} (\bar{\nabla}^k \mathring{\Pi}_{ik}) (\bar{\nabla}_l \mathring{\Pi}^{il}) - 2 \mathring{\Pi}_{ij} \mathring{\Pi}^{ij} + (\mathring{\Pi}_{ij} \mathring{\Pi}^{ij})^2 \right).$$

For  $d \geq 5$  odd, we prove that the obstruction density  $\mathcal{B}$  has a linear highest order term, namely  $\bar{\Delta}^{(d-1)/2} H$  (up to multiplication by a non-zero constant). So:  $\mathcal{B}$  is an analogue of the Willmore invariant; it can be viewed as a fundamental conformal curvature invariant for hypersurfaces; as an obstruction it is an analogue of the Fefferman–Graham obstruction tensor [6]. We see this as follows.

From the algorithm for calculating  $\mathcal{B}$  it is easily concluded that it is a natural invariant (in terms of a background metric), indeed it is given by a formula polynomially involving the second fundamental form and its tangential (to  $\Sigma$ ) covariant derivatives, as well as the curvature of the ambient manifold  $M$  and its covariant derivatives. To calculate the leading term we linearise this formula by computing the infinitesimal variation of  $\mathcal{B}$ . It suffices to consider an  $\mathbb{R}$ -parametrised family of embeddings of  $\mathbb{R}^{d-1}$  in  $\mathbb{E}^d$ , with corresponding defining densities  $\sigma_t$  and such that the zero locus  $\mathcal{Z}(\sigma_0)$  is the  $x^d = 0$  hyperplane (where  $x^i$  are the standard coordinates on  $\mathbb{E}^d = \mathbb{R}^d$ ) so that  $\mathcal{B}|_{t=0} = 0$ . Then applying  $\frac{\partial}{\partial t}|_{t=0}$  (denoted by a dot) we obtain the following:

**Proposition 3.** *The variation of the obstruction density is given by*

$$\dot{\mathcal{B}} = \begin{cases} a \cdot \bar{\Delta}^{(d+1)/2} \dot{\sigma} + \text{lower order terms}, & d-1 \text{ even, with } a \neq 0 \text{ a constant,} \\ \text{non-linear terms,} & d-1 \text{ odd.} \end{cases}$$

This establishes the result, as in this setting the highest order term in the variation of mean curvature is  $\frac{1}{d-1} \bar{\Delta} \dot{\sigma}$ . It also shows that when  $n$  is odd the general formula for  $\mathcal{B}$  may be expressed so that it has no linear term.

Employing methods from tractor calculus [5], and the notion of a holographic formula as introduced in [9] a simple closed formula for the obstruction density in any dimension

can be obtained. Key ingredients of this result are the tractor bundle associated to a conformal structure and the Thomas  $D$ -operator  $D^A$ . Also needed is the projector  $\Sigma_B^A := \delta_B^A - N^A N_B$  from the tractor bundle along  $\Sigma$  to the normal bundle of the normal tractor  $N^A$ . The latter is isomorphic to the tractor bundle of the hypersurface  $\Sigma$  [8] and  $\bar{D}_A$  is its intrinsic Thomas- $D$  operator. For an explanation of these details see Section 4 below as well as [5, 10]. In these terms our result is as follows:

**Theorem 4.** *Let  $\bar{\sigma}$  be a unit conformal defining density. Then, the ASC obstruction density  $\mathcal{B}$  is given by the holographic formula*

$$(-1)^{n+1} \frac{n!(n+2)!}{4} \mathcal{B} = \bar{D}_A \left[ \Sigma_B^A \left( (\bar{I}.D)^n \bar{I}^B - (\bar{I}.D)^{n-1} [X^B K] \right) \right] \Big|_{\Sigma},$$

where  $K := P_{AB} P^{AB}$ ,  $P^{AB} := \hat{D}^A \bar{I}^B$  and  $\bar{I}^A = \hat{D}^A \bar{\sigma}$ .

**Remark 5.** In the above Theorem, the operator  $(\bar{I}.D)^n$  is an example of a sequence of holographic formulæ for tractor twistings of the conformally invariant GJMS operators of [11]. These are a sequence of conformally invariant operators built from powers of the Laplacian with subleading curvature corrections; the simplest examples of these are the Yamabe operator (or conformally invariant wave operator) and Paneitz operator.

The  $d = 3$  invariant (5) is the variation of the Willmore energy  $E = \int_{\Sigma} \ddot{\Pi}_{ij} \ddot{\Pi}^{ij}$ , while in  $d = 4$ , for  $c$  conformally flat, it can be shown that the invariant (6) is the variation of  $E = \int_{\Sigma} \ddot{\Pi}_{ij} \ddot{\Pi}^{jk} \ddot{\Pi}_k^i$ . The naïve conjecture that powers of traces of the trace-free second fundamental form yield integrands for  $d > 4$  energy functionals is bound to fail  $d$  odd because of the leading behavior of the obstruction density given in Proposition 3 (see [13, 7] for a study of conformally invariant  $d = 4$  bending energies). It seems likely that the higher dimensional obstruction densities are variational, therefore it is interesting to speculate whether they are the same as or closely linked to the variations of the submanifold conformal anomalies studied in [12]. A related question is their relevance for entanglement entropy. Recently in [2] variations of the Willmore energy were employed in a study of surfaces maximizing entanglement entropies.

### 3.1 A holographic approach to submanifold invariants

Given a conformal manifold  $(M, c)$  and a section  $\sigma$  of  $\mathcal{E}[1]$  one may construct density-valued conformal invariants that couple the data of the jets of the conformal structure with the jets of the section  $\sigma$ . In the setting of Theorem 2, consider such an invariant  $U$  (say), which uses the section  $\bar{\sigma}$  of the Theorem. Suppose that at every point,  $U$  involves  $\bar{\sigma}$  non-trivially, but uses no more than its  $d$ -jet of  $\bar{\sigma}$ . Then it follows from the first part of Theorem 2 that  $U|_{\Sigma}$  is determined by  $(M, c, \Sigma)$  and so is a conformal invariant of  $\Sigma$ . On the interior the formula for  $U$  as calculated in the scale  $\bar{\sigma}$  (so using  $\bar{\sigma}$  to trivialize the density bundles) is then a regular Riemannian invariant of  $(M, \bar{g})$  (where  $\bar{g} = \bar{\sigma}^{-2} g$ ) which corresponds holographically to the submanifold invariant  $U|_{\Sigma}$ .

## 4 The ideas behind the main proofs

On a conformal manifold  $(M, c)$ , although there is no canonical connection on  $TM$ , there is a canonical linear connection  $\nabla^{\mathcal{T}}$  on a rank  $d + 2$  vector bundle known as the tractor bundle and denoted  $\mathcal{E}^A$  in an abstract index notation. A choice of metric  $g \in c$  determines an isomorphism  $\mathcal{E}^A \xrightarrow{g} \mathcal{E}[1] \oplus T^*M[1] \oplus \mathcal{E}[-1]$ . This connection preserves a metric  $h_{AB}$

on  $\mathcal{E}^A$  that we may therefore use to raise and lower tractor indices. For  $V^A = (\sigma, \mu^a, \rho)$  and  $W^A = (\tau, \nu^a, \kappa)$  this is given by  $h(V, W) = h_{AB}V^AW^B = \sigma\kappa + \mathbf{g}_{ab}\mu^a\nu^b + \rho\tau =: V.W$ . Closely linked to  $\nabla^\mathcal{T}$  is an important, second order conformally invariant operator  $D^A : \mathcal{E}[w] \rightarrow \mathcal{E}^A[w-1]$ ; when  $w \neq 1 - \frac{d}{2}$ , we denote  $\frac{1}{d-2w-2}$  times this by  $\widehat{D}$ , where  $\widehat{D}^A\sigma \stackrel{g}{=} (\sigma, \nabla_a\sigma, -\frac{1}{d}(\Delta + \mathbf{J})\sigma)$ , for the case  $\sigma \in \mathcal{E}[1]$ , and  $2\mathbf{J} = \text{Sc}^g/(d-1)$ . For  $\sigma$  a scale, or even a defining density, we shall write  $I_\sigma^A := \widehat{D}^A\sigma$ , which we call the *scale tractor*. Now  $S(\sigma)$  from above is just  $S(\sigma) = I_\sigma^2 := h_{AB}I_\sigma^AI_\sigma^B$ , so the equation (2) has the nice geometric interpretation  $I_\sigma^2 = 1$  [8], and this is critical for our treatment.

Note that it is essentially trivial to solve (3) for the case  $\ell = 1$ . Theorem 2 is then proved inductively via the following Lemma. The Lemma also yields an algorithm for explicit formulae for the expansion, that we cannot explain fully here, but through this and related results the naturality of  $\mathcal{B}$  can be seen.

**Lemma 6.** *Suppose  $I_\sigma^2 = S(\sigma)$  satisfies (3) for  $\ell = k \geq 1$ . Then, if  $k \neq d$ , there exists  $f_k \in \mathcal{E}[-k]$  such that the scale tractor  $I_{\sigma'}$  of the new defining density  $\sigma' := \sigma + \sigma^{k+1}f_k$  satisfies (3) for  $\ell = k+1$ . When  $k = d$  and  $\sigma' := \sigma + \sigma^{d+1}f$ , then for any  $f \in \mathcal{E}[-d]$ ,*

$$I_{\sigma'}^2 = I_\sigma^2 + \mathcal{O}(\sigma^{d+1}).$$

*Proof - Idea.* First because of the scale tractor definition we have

$$(\widehat{D}\sigma')^2 = I_\sigma^2 + \frac{2}{d}I_\sigma.D(\sigma^{k+1}f_k) + \left[\widehat{D}(\sigma^{k+1}f_k)\right]^2.$$

Tractor calculus identities show that the last term is  $\mathcal{O}(\sigma^{k+1})$ , while  $I_\sigma^2 = 1 + \sigma^k A_k$ . Crucially, the operators  $\sigma$  (acting by multiplication) and  $\frac{1}{I_\sigma^2}I_\sigma.D$  generate an  $\mathfrak{sl}(2)$  [9]. Using standard  $\mathcal{U}(\mathfrak{sl}(2))$  identities, we compute that  $f_k := -dA_k/(2(d-k)(k+1))$  which deals with the  $k \neq d$  cases; the same computation gives the  $k = d$  conclusion.  $\square$

*Proof of Proposition 3 - sketch.* The key idea is that for each  $t$  we can replace  $\sigma_t$  with the corresponding normalised defining density  $\bar{\sigma}_t$  which solves  $I_{\bar{\sigma}_t}^2 = 1 + \bar{\sigma}_t^d \mathcal{B}_{\bar{\sigma}_t}$ , via Theorem 2, while maintaining smooth dependence on  $t$ . Then it is easy to prove that  $\mathcal{B}_{\bar{\sigma}_0}|_{\mathcal{Z}(\bar{\sigma}_{t=0})} = 0$ , while  $\partial(I_{\bar{\sigma}_t}^2)/\partial t|_{t=0}$  is proportional to  $I.D\bar{\sigma}$ . So applying  $\frac{\partial}{\partial t}|_{t=0}$  implies that  $\bar{\sigma}$  solves a linear  $I.D$  boundary problem up to  $\mathcal{O}(\bar{\sigma}^d)$  with obstruction  $\mathcal{B}_{\bar{\sigma}}$ . Using [9, Theorem 4.5] we can easily deduce the conclusion.  $\square$

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## References

- [1] L. Andersson, P. Chrusciel and H. Friedrich, *On the Regularity of solutions to the Yamabe equation and the existence of smooth hyperboloidal initial data for Einstein's field equations*, Commun. Math. Phys. **149**, 587–612 (1992). 1, 3
- [2] A.F. Astaneh, G. Gibbons and S. N. Soludukhin, *What surface maximizes entanglement entropy?*, arXiv:1407.4719. 4
- [3] P. Aviles and R.C. McOwen, *Complete conformal metrics with negative scalar curvature in compact Riemannian manifolds*. Duke Math. J., 56, 395–398 (1988). 1
- [4] P. Aviles and R.C. McOwen, *Conformal deformation to constant negative scalar curvature on non-compact Riemannian manifolds*. J. Differential Geom., 27, 225–239 (1988). 1

- [5] T.N. Bailey, M.G. Eastwood, and A.R. Gover, *Thomas's structure bundle for conformal, projective and related structures*, Rocky Mountain J. Math. **24** (1994), 1191–1217. 3, 4
- [6] C. Fefferman, and C.R. Graham, *The Ambient Metric*, Annals of Mathematics Studies, 178. Princeton University Press, Princeton, NJ, 2012. x+113 pp. 2, 3
- [7] Y. Vyatkin, University of Auckland, Ph.D. thesis, 2013; A.R. Gover and Y. Vyatkin, in progress. 3, 4
- [8] A.R. Gover, *Almost Einstein and Poincaré-Einstein manifolds in Riemannian signature*, J. Geometry and Physics, **60** (2010), 182–204, arXiv:0803.3510 4, 5
- [9] A.R. Gover, and A. Waldron, *Boundary calculus for conformally compact manifolds*, Indiana U.M.J. **63** (2014) 120–163 arXiv:1104.2991. 3, 5
- [10] A.R. Gover, and A. Waldron, in preparation. 2, 4
- [11] C.R. Graham, R. Jenne, Ralph, L. Mason and G. Sparling, *Conformally invariant powers of the Laplacian. I. Existence*, J. London Math. Soc. (2) **46** (1992), 557–565. 4
- [12] C. R. Graham and E. Witten, *Conformal Anomaly Of Submanifold Observables In AdS/CFT Correspondence* Nucl. Phys. **B** 546 (1999) 52–64. 4
- [13] J. Guven, *Conformally invariant bending energy for hypersurfaces*, J. Phys. A: Math. Gen. **38** (2005) 7943–7955. 4
- [14] C. Loewner and L. Nirenberg. *Partial Differential Equations Invariant under Conformal or Projective Transformations* in: contributions to Analysis, Academic Press, New York, 1974. 1